Math 142 Lecture 13 Notes

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1 Lifting Lemmas and the Fundamental Groups of S^n and T^n

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1.1 Lifting lemmas

While proving $\pi_1(S^1, 1) \cong \mathbb{Z}$, we used two lemmas.

Lemma 1.1. *If $\sigma : [0,1] \to S^1$ is a path in S^1 starting at $1 \in S^1$, then there is a unique path $\tilde{\sigma} : [0,1] \to \mathbb{R}$ such that $\tilde{\sigma}(0) = 0$ and $f \circ \tilde{\sigma} = \sigma$, where $f : \mathbb{R} \to S^1$ is $f(x) = e^{2\pi i x}$.*

Proof. Consider an open cover $S^1 = U \cup V$, where $U = S^1 \setminus \{-1\}$ and $V = S^1 \setminus \{1\}$. Then

$$f^{-1}(U) = \mathbb{R} \setminus f^{-1}(\{-1\}) = \bigcup_{n \in \mathbb{Z}} (n - 1/2, n + 1/2),$$

where each interval $(n - 1/2, n + 1/2) \cong U$ via the homeomorphism $f|_{(n-1/2,n+1/2)}$. Similarly,

$$f^{-1}(V) = \bigcup_{n \in \mathbb{Z}} (n, n+1),$$

where each interval $(n, n+1) \cong V$ via the homeomorphism $f|_{(n,n+1)}$. Taking the path $\sigma : [0,1] \to S^1$, we have that since $\{U, V\}$ is an open cover of S^1, $\{\sigma^{-1}(U), \sigma^{-1}(V)\}$ is an open cover of $[0,1]$.

Claim: Since $[0,1]$ is a compact metric space, we can break $[0,1]$ into $[t_0, t_1] \cup [t_1, t_2] \cdots \cup [t_{m-1}, t_m]$, where $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ and $[t_i, t_{i+1}] \subseteq \sigma^{-1}(U)$ or $\sigma^{-1}(V)$. Note that $\sigma(0) = 1 \in U$, so $[t_0, t_1] \subseteq \sigma^{-1}(U)$, or, equivalently, $\sigma([t_0, t_1]) \subseteq U$. We have a homeomorphism $(-1/2, 1/2) \to U$, so define $\tilde{\sigma}(x) = (f|_{(-1/2,1/2)})^{-1}(\sigma(x))$ for $x \in [t_0, t_1]$. Using induction, assume that $\tilde{\sigma}$ is defined on $[t_0, t_1]$. Then $\sigma([t_1, t_{i+1}]) \subseteq U$ or V.

If $\sigma([t_1, t_{i+1}]) \subseteq U$ and $\tilde{\sigma}(t_i) \in (n-1/2, n+1/2)$, define $\tilde{\sigma}(x) = (f|_{(n-1/2,n+1/2)})^{-1}(\sigma(x))$ for $x \in [t_i, t_{i+1}]$. If $\sigma([t_1, t_{i+1}]) \subseteq V$ and $\tilde{\sigma}(t_i) \in (n, n+1)$, define $\tilde{\sigma}(x) = (f|_{(n,n+1)})^{-1}(\sigma(x))$ for $x \in [t_i, t_{i+1}]$. $\square$

Lemma 1.2. *If σ, σ' are paths from 1 to 1 in S^1 with $\sigma \simeq_F \sigma'$ rel $\{0,1\}$, then there exists a unique homotopy $\tilde{F}$ rel $\{0,1\}$ from $\tilde{\sigma}$ to $\tilde{\sigma}'$ such that $f \circ \tilde{F} = F$, where $\tilde{\sigma}, \tilde{\sigma}'$ are the lifts of σ, σ'.*

Proof. The proof of this lemma is similar to that of the previous lemma, so we just provide a sketch. Break the domain of the homotopy into small squares S_i such that $F(S_i)$ is in U or in V, and then define $\tilde{F}$ similarly to how we defined $\tilde{\sigma}$ in the previous lemma. □

For more details, see the proof of lemma 5.11 in the Armstrong textbook.

1.2 The fundamental groups of S^n and T^n

We have shown that $\pi_1(S^1) \cong \mathbb{Z}$. What about the fundamental group of S^n for $n \geq 2$?

Definition 1.1. A space X is *simply connected* if $\pi_1(X, p) \cong 1$.

Theorem 1.1. *Let X be connected, and $X = U \cup V$ open, simply connected, and path-connected. Then for $p \in U \cap V$, $\pi_1(X, p) \cong 1$.*[1]

Proof. We want to show that each path σ in X from p to q is homotopic rel $\{0,1\}$ to $\gamma_1 \cdot \gamma_2 \cdot \gamma_3 \cdots \gamma_m$ for γ_i a path from p to p in U or in V. If so, then $\gamma_i \simeq e_p$ rel $\{0,1\}$ (where $e_p(x) = p$ for all x) for all $i = 1, \dots, m$, so $\sigma \simeq e_p$ rel $\{0,1\}$. $\gamma_i \simeq e_p$ rel $\{0,1\}$ as $\pi_1(U,p) \cong \pi_1(V,p) \cong 1$.

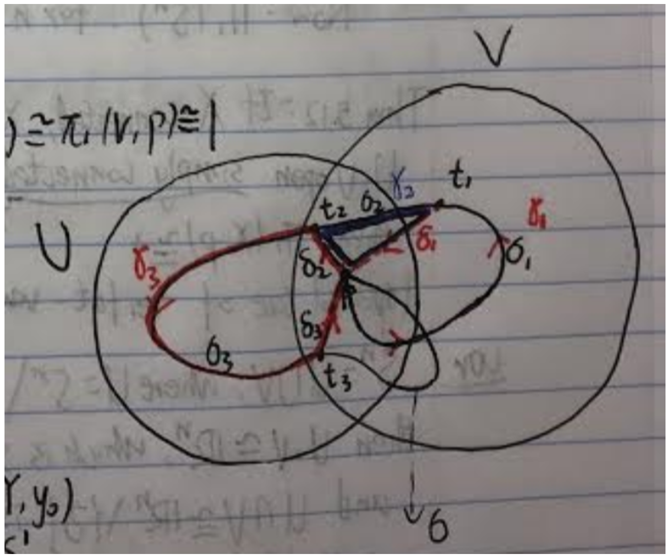
Given $\sigma : [0,1] \to X$, choose $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$ such that $\sigma([t_i, t_{i+1}]) \subseteq U$ or V (as in the lemma before). Let σ_i be the part of σ from $\sigma(t_{i-1})$ to $\sigma(t_i)$. Let δ_i be the path from $\sigma(t_i)$ to p such that

1. δ_i is in U if $\sigma_i(t_i) \in U$,
2. δ_i is in V if $\sigma_i(t_i) \in V$,
3. δ_i is in $U \cap V$ if $\sigma_i(t_i) \in U \cap V$.

So

$$\begin{aligned} \sigma &\simeq \sigma_1 \cdot \delta_1 \cdot \delta_1^{-1} \cdot \sigma_2 \delta_2 \cdot \delta_2^{-1} \cdot \sigma_3 \cdot \delta_3 \cdots \delta_{m-1} \cdot \sigma_m \text{ rel } \{0,1\} \\ &= \gamma_1 \cdot \gamma_2 \cdots \gamma_m. \end{aligned}$$

[1] This is a special case of the Seifert-van Kampen theorem.



For each i, $[\gamma_i] \in \pi_1(U,p)$ or $\pi_1(V,p)$. But $\pi_1(U,p) \cong \pi_1(V,p) \cong 1$, so $\gamma_i \simeq e_p$ rel $\{0,1\}$. So $\sigma \simeq e_p$ rel $\{0,1\}$, and we get that $\pi_1(X,p) \cong 1$. □

Corollary 1.1. $\pi_1(S^n) \cong 1$ *for* $n \geq 2$.

Proof. $S^n = U \cup V$, where $U = S^n \setminus \{\text{north pole}\}$ and $V = S^n \setminus \{\text{south pole}\}$. Then $U, v \cong \mathbb{R}^n$, which is simply connected, and $U \cap V \cong \mathbb{R}^n \setminus \{0\}$ is path-connected for $n \geq 2$. We can then apply the theorem. □

Theorem 1.2. $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Example 1.1. Let $T^n = S^1 \times \cdots \times S^1$ be the n-dimensional torus. Then $\pi_1(T_n) \cong \mathbb{Z}^n$.